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LETTER TO THE EDITOR

Classification of quantum group structures on the group $GL(2)$

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Abstract. All quantum group structures on the group $GL(2)$ are classified. In addition to the known quantum groups $GL_{q,p}(2)$ and $GL_{h,h'}(2)$, there exists one exceptional new quantum group $GL_{h,q}(2)$.

The problem of classifying all quantum group structures on a given Lie group is, in general, a hopeless one even in the quasiclassical limit. Exceptions are either special type groups, such as complex semisimple (Belavin and Drinfel'd 1982) or the group of formal diffeomorphisms of the line (Kupershmidt and Stoyanov 1992), or else groups of small dimension. The group $GL(2)$, one of the most popular quantum objects, has dimension 4 which, though not small, is not too large either. Moreover, all quantum structures on $GL(2)$ can be classified.

The known quantum structures on $GL(2)$ are of two types: $GL_{q,p}(2)$ (Manin 1991) and $GL_{h,h'}(2)$ (Aghamohammadi 1993). They are described by the following relations on the elements a, b, c, d of a 2×2 quantum matrix $M=\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$:

$$\begin{aligned} &ac=q^{-1}ca \qquad bd=q^{-1}db \qquad ab=p^{-1}ba \\ &cd=p^{-1}dc \qquad bc=q^{-1}pcb \qquad [a,d]=(q^{-1}-p)cb \end{aligned} \tag{1}$$

$$\begin{aligned} &[a,c]=hc^2 \qquad [b,d]=h(d^2+cb-ad+hcd) \\ &[c,d]=-h'c^2 \qquad [a,d]=hcd-h'ca \\ &[b,c]=hdc+h'ca \qquad [a,b]=h'(ad-bc-a^2+h'ac). \end{aligned} \tag{2}$$

Each of these quantum groups satisfies the PBW property and has a multiplicative quantum determinant:

$$\mathrm{Det}_{q,p}(M)=da-pcb=ad-p^{-1}bc \tag{3}$$

$$\mathrm{Det}_{h,h'}(M)=ad-bc+h'ac=da-cb-h'ca. \tag{4}$$

Multiplicativity means that the identity

$$\mathrm{Det}(M_1M_2)=\mathrm{Det}(M_1)\,\mathrm{Det}(M_2)$$

is satisfied for any two quantum matrices M_1 and M_2 with mutually commuting entries. The quantum determinants $\mathrm{Det}_{q,p(M)}$ (3) and $\mathrm{Det}_{h,h'}$ (4) are central only when: (i) $p=q$, in which case one recovers the quantum group $GL_q(2)$ (Drinfel'd 1986a); and (ii) $h'=h$, when one obtains the quantum group $GL_h(2)$ (Zakrzewski 1991). In (Kupershmidt 1992) I proved that $GL_q(2)$ and $GL_h(2)$ are the only, up to isomorphism, quantum group structures on $GL(2)$ with a central quantum determinant. (At the time I

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was not aware of the Zakrzewski paper which was subsequently brought to my attention by P Kulish.) If the quantum determinant is allowed not to be central, are there any other quantum group structures on $GL(2)$, beside $GL_{q,p}(2)$ (1) and $GL_{h,h'}(2)$ (2)?

There exists just one such, $GL_{h,q,h'}(2)$, given by the relations

$$
\begin{aligned}
&[a, c]=0 \qquad [b, d]=0 \\
&ab=qba-ha^2-h'b^2+h(ad-qbc+hac+h'bd) \\
&cd=qdc-hc^2-h'd^2+h'(ad-qbc+hac+h'bd) \\
&cb=qbc-hac-h'bd \qquad [a, d]=(q-1)bc-hac-h'bd
\end{aligned}
\tag{5}
$$

with the multiplicative quantum determinant

$$
\begin{aligned}
\mathrm{Det}_{h,q,h'}(M)&=ad-qbc+hac+h'bd \\
&=da-q^{-1}cb-q^{-1}hca-q^{-1}h'db.
\end{aligned}
\tag{6}
$$

These formulae can be arrived at via the following route, parallel to the one in (Kupershmidt 1992). We first start with the quasiclassical picture, i.e. with multiplicative Poisson brackets on the group $GL(2)$. For a vector space V every multiplicative Poisson bracket on $GL(V)$ is quadratic and is induced by a pair of quadratic Poisson structures on V and $\Lambda^1(V)$. In our case V is two-dimensional, and the corresponding quadratic Poisson structures on V and $\Lambda^1(V)$ can be brought into the form

$$\{x, y\}=y(rx+sy) \tag{7}$$

$$\{\xi, \xi\}=u\xi\eta \qquad \{\xi, \eta\}=v\xi\eta \qquad \{\eta, \eta\}=w\xi\eta \tag{8}$$

where r, s, u, v, w are (even) constants. The multiplicative Poisson brackets induced on $GL(2)$ are:

$$
\begin{aligned}
&\{a, c\}=c(ra+sc) \\
&\{a, b\}=-vab-wb^2/2-u(a^2+bc-ad)/2 \\
&\{b, d\}=(rb+sd)d+s(bc-ad) \\
&\{c, d\}=-vcd-uc^2/2-w(d^2+bc-ad)/2 \\
&\{a, d\}=(r-v)bc+scd-uac/2-wbd/2 \\
&\{b, c\}=(r+v)bc+scd+uac/2+wbd/2.
\end{aligned}
\tag{9}
$$

The Jacobi identities divide the parameter-space $\{r, s, u, v, w\}$ into the following three regions:

(A) $r=s=0$; u, v, w are arbitrary.
(B) $r\neq 0$. A linear change of basis in V can make $s=0$. Then $u=w=0$; v is arbitrary.
(C) $r=0$, $s\neq 0$. Then $v=w=0$; u is arbitrary.

Quantizing these three alternatives, we get:

$$xy = yx \tag{10a}$$

$$\xi^2 = h\xi\eta \qquad \eta\xi = -q\xi\eta \qquad \eta^2 = h'\xi\eta \tag{10b}$$

which leads to formulae (5);

$$xy = q^{-1}yx \tag{11a}$$

$$\xi^2 = 0 \qquad \eta\xi = -p^{-1}\xi\eta \qquad \eta^2 = 0 \tag{11b}$$

which leads to formulae (1); and

$$[x, y] = hy^2 \tag{12a}$$

$$\xi^2 = h'\xi\eta \qquad \eta\xi = -\xi\eta \qquad \eta^2 = 0 \tag{12b}$$

which leads to formulae (2). The two expressions for the quantum determinant, given by the formulae (6), (3) and (4), result from the standard definition

$$\tilde{\xi}\tilde{\eta} = \mathrm{Det}(M)\xi\eta \qquad \tilde{\eta}\tilde{\xi} = \mathrm{Det}(M)\eta\xi \tag{13}$$

where

$$\begin{pmatrix} \tilde{\xi} \\ \tilde{\eta} \end{pmatrix} = M\begin{pmatrix} \xi \\ \eta \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} \xi \\ \eta \end{pmatrix}.$$

It remains to settle the PBW problem for the quantum group $GL_{h,q,h'}(2)$ (5). Since there exists no ordering of the generators a, b, c, d which makes the relations (5) of descending type, the diamond Lemma (Bergman 1978) cannot be used. (This is clear even on the level of the space $\Lambda^1(V)$ (12*b*).) In principle, one can use the criterion of (Drinfel'd 1986b), see also (Manin 1988), but the Drinfel'd criterion for quadratic commutation relations will be too cumbersome to apply. Instead, by a linear change of basis in V one can make h' vanish: Indeed, if $h = h' = 0$ then the relations (10) are a particular case of the relations (11); if $h = 0, h' \neq 0$ then we get $h \neq 0, h' = 0$ by interchanging the basis vectors of V and $\Lambda^1(V)$; finally, of $h \neq 0$ and $h' \neq 0$, we apply the matrix

$$\begin{pmatrix} 1 & 0 \\ 1 & t \end{pmatrix} \tag{14}$$

to the vectors $\binom{x}{y}$ and $\binom{\xi}{\eta}$. Choosing t to be a (non-zero) root of the quadratic equation

$$h't^2 + (1-q)t + h = 0 \tag{15}$$

we achieve $\eta^2 = 0$.

With the choice $h' = 0$ from now on, we can rewrite the commutation relations (5) for the quantum group $GL_{h,q}(2)$ as:

$$\begin{aligned} &ba = q^{-1}a(b+ha) - hda + q^{-1}hc(b+ha) \qquad bd = db \\ &bc = q^{-1}c(b+ha) \qquad ad = da + (1-q^{-1})cb - q^{-1}hca \\ &ac = ca \qquad dc = q^{-1}c(d+hc). \end{aligned} \tag{16}$$

These relations satisfy the decreasing order condition for the ordering $b>a>d>c$. The straightforward application of the diamond Lemma and the identity

$$[b+ha, d+hc]=0 \tag{17}$$

show that the PBW property is satisfied for the quantum group $GL_{h,q}(2)$ (and, therefore, for the quantum group $GL_{h,q,h'}(2)$). The quantum determinant formulae (6) now become

$$\mathrm{Det}_{h,q}(M)=ad-qbc+hac=da-q^{-1}cb-q^{-1}hca \tag{18a}$$

which can also be written as

$$\mathrm{Det}_{h,q}(M)=ad-cb=da-bc. \tag{18b}$$

We also have the formulae for the adjugate matrices M_r and M_l of M:

$$MM_r=M_lM=\mathrm{Det}_{h,q}(M)1 \tag{19}$$

$$M_r=\begin{pmatrix} d+hc & -q^{-1}(b+ha)+hd+h^2c \\ -qc & a-hqc \end{pmatrix} \tag{20a}$$

$$M_l=\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}. \tag{20b}$$

Formulae (18b) and (20b) are somewhat mystifying, but they suggest how, using induction on n, one can find analogues of the quantum group $GL_{h,q}(2)$ for the case of $GL(n)$ with $n\geqslant 3$.

Further properties of the quantum group $GL_{h,q}(2)$ follow.

(A) Formulae (16) show that the element c is normalizing, so that setting $c=0$ one arrives at the upper-triangular subgroup $\Delta^+_{h,q}(2)$ of $GL_{h,q}(2)$ consisting of matrices $\left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right)$ with the commutation relations

$$[d,a]=[d,b]=0 \qquad ba=q^{-1}a(b+ha)-hda. \tag{21}$$

(B) The quantum determinant $D=\mathrm{Det}_{h,q}(M)$ is also normalizing:

$$\begin{aligned} &aD=D(a-hqc) \qquad cD=qDc \qquad dD=D(d+hc) \\ &bD=D[q^{-1}(b+ha)-hd-h^2c] \end{aligned} \tag{22}$$

$$\begin{aligned} &Da=(a+hc)D \qquad Dc=q^{-1}cD \qquad Dd=(d-q^{-1}hc)D \\ &Db=(qb-ha+qhd-h^2c)D. \end{aligned} \tag{23}$$

Formulae (23) are inverse to formulae (22). The latter can be obtained from the identity

$$M_lD=M_l(MM_r)=(M_lM)M_r=DM_r \tag{24}$$

which follows from formula (19).

(C) Using formulae (22), (23), one can show that

$$M^{-1}\in GL_{-h,q^{-1}}(2) \tag{25}$$

$$\mathrm{Det}_{-h,q^{-1}}(M^{-1})=[\mathrm{Det}_{h,q}(M)]^{-1}. \tag{26}$$

This suggests that, more generally,

$$M^k\in GL_{kh,q^k}(2) \qquad k\in Z \tag{27}$$

similar to the cases of $GL_q(2)$ and $GL_h(2)$. Interestingly, this is not true. Restricting from $GL_{h,q}(2)$ onto the upper-triangular subgroup $\Delta^{+}_{h,q}(2)$ (21) one quickly finds that for (27) to hold one must have

$$q=1. \tag{28}$$

(D) The same special value $q=1$ can be arrived at via a different route. Let

$$S=\begin{pmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{pmatrix}$$

be a matrix whose entries commute with those of M. Then a linear functional on S ('Trace'), invariant under the action

$$S \to M^{-1}SM \tag{29}$$

exists if and only if $q=1$, in which case one has

$$\mathrm{Tr}(S)=s_{11}+s_{22}+hs_{21}. \tag{30}$$

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References

Aghamohammadi A 1993 *Mod. Phys. Lett.* A **8** 2607

Belavin A and Drinfel'd V G 1982 *Funct. Anal. Appl.* **16** 159; **17** 220

Bergman G 1978 *Adv. Math.* **29** 178

Drinfel'd V G 1986a *Proc. Int. Congr. Math. (Berkeley)* p 798

—— 1986b *Mathematica, Physica, and Functional Analysis* (Kiev: Naukova Dumka) p 25

Kupershmidt B A 1992 *J. Phys. A: Math. Gen.* **25** L1239

Kupershmidt B A and Stoyanov O S 1992 Classification of all Poisson–Lie structures on an infinite-dimensional jet group *Preprint*

Manin Yu I 1988 *Quantum Groups and Non-Commutative Geometry* (Montreal: CRM)

Manin Yu I 1991 *Topics in Noncommutative Geometry* (Princeton, NJ: Princeton University Press)

Zakrzewski S 1991 *Lett. Math. Phys.* **22** 287